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CONSTRUCTION OF GREEN'S FUNCTION FOR AN ANISOTROPIC HEREDITARY-ELASTIC MEDIUM *

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In order to reduce the static problems of deformation of solids to integral equations, it is necessary to construct a Green's function describing the reaction of the medium in question to a concentrated unit force.

Let us consider an anisotropic hereditary-elastic medium defined by the equation

$$\sigma_{ij}(t) = c_{ijkl}\varepsilon_{kl}(t) + b^*_{ijkl}\int_0^t \frac{\varepsilon_{kl}(\tau)\,d\tau}{(t-\tau)^\alpha} \quad (1)$$

Here σ_{ij} denote the stresses, ε_{ij} the deformations, the tensors c_{ijkl} and b^*_{ijkl} describe the elastic and hereditary properties of the deformed material, and t is time. The scalar parameter α varies over the interval $0<\alpha<1$. The feasibility of using the tensor b^*_{ijkl} only to describe the anisotropy of the hereditary properties is confirmed by, for example, the experimental results of /1/.

When $b^*_{ijkl}=0$, equation (1) becomes a defining equation for an anisotropic medium without hereditary properties, and the Green's function for this case has been constructed in /2, 3/.

Let us pass from $\sigma_{ij}(t)$ and $\varepsilon_{ij}(t)$ to their Laplace transforms $\bar\sigma_{ij}(p)$ and $\bar\varepsilon_{ij}(p)$. Then (1) becomes ($\Gamma(\beta)$ is the gamma function)

$$\bar\sigma_{ij} = a_{ijkl}\bar\varepsilon_{kl}$$
$$a_{ijkl} = c_{ijkl} + b_{ijkl}p^{-\beta}, \quad b_{ijkl} = \Gamma(\beta)\, b^*_{ijkl}, \quad \beta = 1-\alpha$$

Clearly, the transform of the Green's function satisfies the following expression analogous to the expression for an anisotropic medium without heredity

$$\bar u_{ij}(\mathbf{x},\mathbf{y},p) = \frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|}\oint_{|\zeta|=1} K_{ij}^{-1}(\zeta)\,ds, \quad K_{ij} = a_{ikjl}\zeta_k\zeta_l \quad (2)$$

Here $\mathbf{y}$ and $\mathbf{x}$ are the radius vectors of the points, at which the force is applied and the reaction of the medium sought. The contour of integration lies in the plane normal to the vector $\mathbf{x}-\mathbf{y}$.

Although the relation (2) solves, in principle, the problem of constructung the Green's function for a hereditary-elastic medium, it is of little use for practical computations since its inversion in the form given can, so far, be only carried out by numerical methods.

Let us compute the inverse transform of $\bar u_{ij}(p)$, i.e. let us invert the expression (2). We write

$$c_{ij} = c_{ikjl}\zeta_k\zeta_l, \quad b_{ij} = b_{ikjl}\zeta_k\zeta_l, \quad \lambda = p^{-\beta}, \quad a_{ij} = c_{ij} + b_{ij}p^{-\beta}, \quad d_{ij} = a_{ij}^{-1}$$

The matrix d_{ij} is an inverse of a_{ij}, therefore we have (δ_{ij} is the Kronecker delta)

$$d_{il}a_{lj} = \delta_{ij} \quad (3)$$

We seek d_{ij} in the form

$$d_{ij} = d_{ij}^{(0)} + \lambda d_{ij}^{(1)} + \lambda^2 d_{ij}^{(2)} + \lambda^3 d_{ij}^{(3)} + \ldots \quad (4)$$

Substituting (4) into (3) and equating to zero the consecutive cofactors accompanying the like powders of λ, we obtain the following recurrent relations for computing the matrices $d_{ij}^{(n)}$. As a result we have

$$\bar u_{ij}(\mathbf{x},\mathbf{y},p) = \frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|}\oint_{|\zeta|=1}\{d_{ij}^{(0)} + p^{-\beta}d_{ij}^{(1)} + p^{-2\beta}d_{ij}^{(2)} + \ldots\}\,ds$$

The series under the symbol of contour integration converges uniformly, provided that ($\|\cdot\|$ is the norm of the matrix)

$$|p| > (\|d_{ij}^{(0)}\|\,\|b_{ij}\|)^{1/\beta}$$

Assuming now that the straight line, parallel to the imaginary axis, along which the Mellin integral is computed, passes through the domain of uniform convergence of the series, we obtain the required expression for the Green's function

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$$u_{ij}(\mathbf{x},\mathbf{y},t)=\frac{1}{8\pi^2|x-y|}\left\{D_{ij}^{(0)}+D_{ij}^{(1)}\frac{t^{-\alpha}}{\Gamma(\beta)}+D_{ij}^{(2)}\frac{t^{1-2\alpha}}{\Gamma(2\beta)}+D_{ij}^{(3)}\frac{t^{2-3\alpha}}{\Gamma(3\beta)}+\ldots\right\},\quad D_{ij}^{(n)}=\oint_{|\zeta|=1}d_{ij}^{(n)}(\zeta)\,ds \tag{5}$$

It can be shown that the series (5) defining the Green's function converges uniformly in t over any interval $0\leqslant t<T$ where T can be arbitrarily large. Obviously, we have

$$\left\|D_{ij}^{(0)}+\sum_{k=1}^{\infty}D_{ij}^{(k)}\frac{t^{k\beta-1}}{\Gamma(k\beta)}\right\|\leqslant D\left\{1+\sum_{k=1}^{\infty}q^k\frac{t^{k\beta-1}}{\Gamma(k\beta)}\right\},\qquad D=\|D_{ij}^{(0)}\|,\quad q=D\|b_{ij}\| \tag{6}$$

Let us now discard in (6) the terms of negative power in t. The corresponding values of the index k satisfy the inequality $k\leqslant k_0=[1/\beta]$ (the square brackets denote the integral part of the real number $1/\beta$ obtained by rounding-off upwards; there is a finite number of such terms). Then we have

$$\sum_{k_0+1}^{\infty}\frac{q^k t^{(k\beta-1)}}{\Gamma(k\beta)}\leqslant\sum_{k_0+1}^{\infty}A_k,\quad A_k=q^k\frac{T^{(k\beta-1)}}{\Gamma(k\beta)} \tag{7}$$

To confirm the convergence of the series (7) we use the asymptotic expression for $\Gamma(k\beta)$ for large k given by the Stirling's formula, and apply the D'Alembert's test. We have

$$\frac{A_{k+2}}{A_{k+1}}\leqslant qT^{\beta}\frac{[k\beta/e]^{k\beta}\sqrt{k}}{[(k+1)\beta/e]^{(k+1)\beta}\sqrt{k+1}}\underset{k\to\infty}{\longrightarrow}0$$

which implies the convergence of the series in question. Thus, if a concentrated force $F_j(t)$ is applied at the point $\mathbf{y}$ of the medium described by (1) then, using (5) we obtain the following expression for the displacement at the point $\mathbf{x}$ caused by this force:

$$u_i(\mathbf{x},\mathbf{y},t)=\frac{1}{8\pi^2|\mathbf{x}-\mathbf{y}|}\left\{D_{ij}^{(0)}F_j(t)+\sum_{k=1}^{\infty}\frac{D_{ij}^{(k)}}{\Gamma(k\beta)}\int_0^t\tau^{k\beta-1}F_j(t-\tau)\,d\tau\right\}$$

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